ALGORITHM 60
ROMBERG INTEGRATION

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```
real procedure rombergintegr (fct, ℓgr, rgr, ord)  ;
  value ℓgr, rgr, ord  ;
  real ℓgr, rgr;  integer ord  ;  real procedure fct  ;
  comment rombergintegr is the value of the integral of the
    function fct between the limits ℓgr and rgr, calculated by the
    algorithm of Romberg with an error term of the order
    2×ord+2, ord≧0 Computation time will roughly be doubled
    when ord is increased by 1;
  begin
    real array t[1 : ord+1];
    real  ℓ,  u,  m  ;
    integer f, h, j, n  ;
    ℓ  :=  rgr−ℓgr  ;
    t[1]  :=  (fct(ℓgr)+fct(rgr))/2  ;
    n := 1  ;
    for h := 1 step 1 until ord do
      begin u  :=  0  ;
        m  :=  ℓ/(2×n)  ;
        for j := 1 step 2 until 2×n−1 do
          u := u+fct(ℓgr+j×m)  ;
        t[h+1]  :=  (u/n+t[h])/2  ;
        f  := 1  ;
        for j := h step − 1 until 1 do
          begin f  :=  4×f  ;
            t[j]  :=  t[j+1]+(t[j+1]−t[j])/(f−1)
          end  ;
        n := 2×n
      end  ;
    rombergintegr := t[1]×ℓ
  end
```

CERTIFICATION OF ALGORITHM 60
ROMBERG INTEGRATION (F. L. Bauer, *Comm. ACM*, June, 1961)

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* Work supported by the U. S. Atomic Energy Commission.

This procedure was translated to the ACT III compiler language for the Royal Precision LGP-30 computer. This system provides 7+ significant decimal digits. The program was used to integrate x^n between the limits 0.01 and 1.1, and between the limits 1.1 and 0.01. The results in Table I were obtained. The pole at 0 for negative n affords a test of the reliability of the method when the higher derivatives of the integrand are large. The agreement between integrations in the forward and backward directions is an indication of the effects of round-off error.

It is apparent that the procedure gives results well within the noise level for the positive powers, and that even the effect of a closely adjacent singularity for the negative powers can be overcome.

The flexibility of the algorithm would be improved by adding to the formal parameters a procedure, check, to decide if sufficient

TABLE I. Integration of $\int_{0.01}^{1.1} x^n dx$ and $\int_{1.1}^{0.01} x^n dx$

n	0	+12	+12	−1
True Value	1.0900000	.26555932	−.26555932	4.7004831
Order 1	1.0899997	.57076812	−.57076842	19.641113
Order 2	1.0899997	.30614608	−.30614626	10.656923
Order 5	1.0899991	.26555693	−.26555818	4.9017590
Order 10				4.7002345

n	−1	−5	−5
True Value	−4.7004831	$.25000000\times10^8$	-18.166667×10^8
Order 1	−19.641125	18.166655×10^8	$-.25000000\times10^8$
Order 2	−10.656929	8.4777719×10^8	-8.4777766×10^8
Order 5	−4.9017805	1.0408634×10^8	-1.0408640×10^8
Order 10	−4.7004402	$.25000715\times10^8$	$-.25000727\times10^8$
Order 12		$.24999291\times10^8$	$-.25001311\times10^8$

accuracy had been obtained without carrying through the entire iteration. A possible form for this procedure would be:

```
procedure check (t1, t2, f, exit);
  real t1, t2;
  label exit;
  integer f;
begin if abs ((t2 − t1) × f) / t1 < tolerance ∧ f > minimum order
  then go to exit end.
```

The global variables tolerance, which is the maximum relative difference between approximations of increasing order, and the minimum acceptable order should be selected by the programmer for the exigencies of the problem. A check of this sort is clearly not as sound as an a priori estimate of the necessary order, but is frequently an acceptable expedient.

The Romberg quadrature algorithm is analyzed in the following references:

Romberg, W. Vereinfachte numerische Integration. *Det Kongelinge Norske Videnskaber Selskab Forhandlinger 28*, (1955), 30-36.

Stiefel, E., and Rutishauser, H. Remarques concernant l'integration numerique. *Comptes Rendus Acad. Scil (Paris) 252*, (1961), 1899-1900.

CERTIFICATION OF ALGORITHM 60
ROMBERG INTEGRATION (F. L. Bauer, *Comm. ACM*, June 1961)

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Since August 1961, the Rombert Integration has been successfully applied in Fortran language to various problems on an IBM 1620. Due to its elegant method and the memory saving features, the Romberg Integration has succeeded other methods in our program library, e.g., the Newton-Cotes integration of order 10.

Reference is made to Stiefel, *Numerische Mathermatik* (Teubner Verlag. Stuttgart). Stiefel discusses in his book various methods of numerical integration including the Romberg algorithm.

CERTIFICATION OF ALGORITHM 60 ROMBERG INTEGRATION (F. L. Bauer, *Comm. ACM*, June 1961)

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REMARK ON ALGORITHM 60 [D1] ROMBERG INTEGRATION [F. L. Bauer, *Comm. ACM 4* (June 1961) 255; *5* (Mar. 1962), 168; *5* (May 1962), 281]

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* Work supported by the U. S. Atomic Energy Commission.

The Romberg integration algorithm has been used with great success by many groups [1, 2], and appears to be among the most generally reliable quadrature methods available. It is, therefore, worth pointing out that it is not entirely foolproof, and that a significant class of integrands exists for which the extrapolated values are poorer estimates of the integral than the corresponding trapezoidal sums.

The validity of the Romberg procedure depends upon the possibility of expanding the error of the trapezoidal rule in powers of h^2, where h is the stepsize. One expansion of this type is the Euler-Maclaurin sum formula. An alternative expression may be obtained from the Fourier series expansion. The coefficients of h^{2r} in the Euler Maclaurin formula are proportional to the difference of the values of the $(2r+1)$-th derivative at the two ends of the range. Thus, any integral for which the odd derivatives of the integrand either vanish or are equal at the limits will not be improved by Romberg extrapolation. Among the common examples of such integrals are integrals of periodic functions over a period and integrals for which the derivatives vanish at both limits. An example of the last type is the integral approximation to the modified Hankel function [3], $e^x K_p(x) = \int_0^L e^{x(1-\cosh t)} \cosh (pt) dt$, where L is taken so large that the contribution of the integral from L to ∞ may be neglected. Several other examples are given under the heading "Exceptional cases" by Bauer, Rutishauser and Stiefele [7]. This paper is among the most extensive discussions of the Romberg method in English.

The algorithm also fails when the expansion of the error term contains other powers of h along with the even ones. Rutishauser [4] discusses estimating integrals of the form $\int_0^a f(x)\, dx = \int_0^a (\varphi(x)/\sqrt{x})\, dx$. If such integrals are estimated by the trapezoidal rule, assigning the value 0 to $f(0)$, the error may be expressed in the form $\sum c_k h^{2k} + \sqrt{h} \sum d_k h^k$. Although the standard Romberg extrapolation fails when applied to this sequence of estimates, Rutishauser presents a modified procedure which is effective.

The extrapolation is also invalid when the integrand is discontinuous, although this exception is trivial from the computational standpoint.

It has also been pointed out [5, 6] that the Romberg procedure may amplify round-off errors. The losses, while significant, do not appear prohibitive for most applications.

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