ALGORITHM 66
INVRS

JOHN CAFFREY
Director of Research, Palo Alto Unified School District, Palo Alto, California

```
procedure Invrs (t) size : (n); value n; real array t; inte-
  ger n;
comment Inverts a positive definite symmetric matrix t, of
order n, by a simplified variant of the square root method. Re-
places the n(n+1)/2 diagonal and superdiagonal elements of t
with elements of t⁻¹, leaving subdiagonal elements unchanged.
Advantages: only n temporary storage registers are required, no
identity matrix is used, no square roots are computed, only n
divisions are performed, and, as n becomes large, the number of
multiplications approaches n³/2;
begin integer i, j, s; real array v[l:n−1]; real y, pivot;
for s := 0 step 1 until n−1 do
begin pivot := 1.0/t[1,1];
    begin pivot := 1.0/t[1,1];
                    comment If t[1,1] ≦ 0, t is not positive defi-
                    nite;
        for i := 2 step 1 until n do v[i−1] := t[1, i];
        for i := 1 step 1 until n−1 do
            begin t[i,n] := y := −v[i] × pivot;
            for j := i step 1 until n−1 do
            t[i, j] := t[i + 1, j +1] + v[j] × y
            end;
        t[n,n] := −pivot
    end;
                comment At this point, elements of t⁻¹ occupy
                the original array space but with signs reversed,
                and the following statements effect a simple re-
                flection;
    for i := 1 step 1 until n do
    for j := i step 1 until n do t[i,j] := −t[i,j]
end Invrs
```

CERTIFICATION OF ALGORITHM 66
INVRS (J. Caffrey, Comm. ACM, July 1961)

B. RANDELL, C. G. BROYDEN.
Atomic Power Division, The English Electric Company, Whetstone, England.

INVRS was translated using the DEUCE ALGOL Compiler, and needed the following correction.

The repeat of the line,

begin pivot := 1.0/t[1, 1];

was deleted.

The compiled program, which used a 20 bit mantissa floating point notation, was tested using Wilson's matrix

5	7	6	5
7	10	8	7
6	8	10	9
5	7	9	10

and gave results

67.9982	−40.9991	−16.9995	9.9997
−40.9991	24.9995	9.9997	−5.9998
−16.9995	9.9997	4.9998	−2.9999
9.9997	−5.9998	−2.9999	1.9999

(The output routine completed the symmetric matrix)

INVRS will in fact invert non-positive symmetric matrices, the only restriction appearing to be that the leading minors of the matrix must be non-zero. The variable T[1, 1] takes as its successive values ratios of the (r + 1)th to the r th leadng minors of the matrix, and if it becomes zero the variable 'pivot' cannot be computed.

The following matrix, for which the successive values of T[1, 1] were +2, −2, −1, −0.6, +5 gave results correct to one unit in the fifth significant figure.

2	−3	1	−1	4
−3	2	−4	3	−2
1	−4	−3	2	4
−1	3	2	−2	−3
4	−2	4	−3	2

CERTIFICATION OF ALGORITHM 66
INVRS (J. Caffrey, *Comm. ACM*, July 1961)

JOHN CAFFREY
Palo Alto Unified School District, Palo Alto, California

INVRS was translated using the Burroughs 220 Algebraic Computer (BALCOM) at Stanford University, using 8-digit floating-point arithmetic. The misprint noted by Randell and Broyden (*Comm. ACM*, Jan. 1962, p. 50) was corrected, and the same example (Wilson's 4 × 4 matrix) was used as a test case. The resulting inverse was:

68.0000	−41.0000	−17.0000	10.0000
	25.0000	10.0000	−6.0000
		5.0000	−3.0000
			2.0000

It may also be useful to note that the determinant of the matrix may be obtained as the successive product of the pivots. That is, if t_i (= $T(1, 1)$) is the ith pivot of a matrix of order n,

$$\text{determinant} = \prod_i^n t_i .$$

For the above *input* example,

determinant = 1.0

Randell and Broyden's observation concerning the *apparent* limitation of INVRS to positive definite cases is correct: That is, any nonsingular real symmetric matrix (positive, indefinite, or negative) may be inverted using this algorithm. The original INVRS should therefore be modified as follows:

if pivot = 0 **then go to** singular;

Randell and Broyden's second example (of order 5) was also used as a test case, with the resulting inverse:

−.0000	.9999	.0000	.0000	.9999
	1.5333	−.7333	−.1333	.7999
		−.8666	−1.0666	−.5999
			−1.4666	−.1999
				.2000

determinant = −14.999999

An attempt to invert the *inverse* of the 4 × 4 segment of the Hilbert matrix, as presented by Randell (*Comm. ACM*, Jan. 1962, p. 50), yielded the following results:

.9999	.4999	.3333	.2499
	.3333	.2499	.1999
		.1999	.1666
			.1428

determinant = 6048020.6