

# Algorithms

H. J. WEGSTEIN, Editor

## ALGORITHM 77 INTERPOLATION, DIFFERENTIATION, AND INTEGRATION

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**real procedure** AVINT (nop, jt, xarg, xlo, xup, xa, ya);  
**value** nop, jt, xarg, xlo, xup; **real** xarg, xlo, xup;  
**integer** nop, jt; **real array** xa, ya;

**comment** This procedure will perform interpolation, differentiation, or integration operating upon functions of one variable which over part or all of the interval of interest are adequately described by a di-parabolic fit.

The routine was originally programmed as an open subroutine for the IBM 704 in FORTRAN II and occupied 323 memory locations. It is based upon a Lagrange interpolation scheme specialized for averaged second order parabolas. The technique finds the slope of a function numerically defined at points 1, 2, 3 and 4 by fitting a parabola through the points 1, 2, 3, and another parabola through the points 2, 3, and 4. The slope then, at point 2, is the average analytical derivative of the two parabolas, i.e. the coefficients of the parabola through points 1, 2 and 3 ( $a_1x_2^2 + b_1x_2 + c_1$ ) and the coefficients of the parabola through points 2, 3, and 4 ( $a_2x_2^2 + b_2x_2 + c_2$ ) are determined by applying Lagrange's equations as shown below. The arithmetic mean of these coefficients  $a = (a_1 + a_2)/2$ ,  $b = (b_1 + b_2)/2$ ,  $c = (c_1 + c_2)/2$  are used to supply the slope in the interval from 2 to 3, namely  $(2ax + b)$ .

The interpolation is calculated in similar fashion, except the final formula is that a parabola ( $ax^2 + bx + c$ ).

The integration is performed likewise by a curve fitting process, e.g. the integral between any two points say 2 and 3 is the average integral of the two parabolas between the independent coordinate limits for points 2 and 3. The averaging process is done for each interval along the abscissa as the results obtained are accumulated to evaluate the definite integral.

Applying Lagrange's equations, the coefficients a, b, and c may be found by defining:  $T_j = y_j / \prod_{i=1, i \neq j}^n (X_j - X_i)$  where  $y = f(x)$ ,  $n = 3$ ,  $j = 1, 2, \dots, n$ , then  $a = \sum_{i=1}^n T_i$ ,  $b = \sum_{i=1}^n T_i \sum_{j=1, j \neq i}^n X_j$ ,  $c = \sum_{i=1}^n T_i \prod_{j=1, j \neq i}^n X_j$ ;

**begin real** ca, cb, cc, a, b, c, syl, syu, term1, term2, term3, da, dif, sum;

**integer** jm, js, jul, ia, ib;  
**start:** **switch** alpha := L1, L11, L12; **switch** beta := L9, L5, L6;  
**switch** gamma := L10, L11; **switch** delta := L8, L8, L13;

**comment** For interpolation, differentiation or integration set  
jt = 1, 2, or 3 respectively;

**go to** alpha [jt];  
L1: **if** xarg  $\geq$  xa [nop] **then go to** L2;  
**if** xarg  $\geq$  xa [nop-1] **then go to** L2;  
**if** xarg  $\leq$  xa [1] **then go to** L3;  
**if** xarg  $\leq$  xa [2] **then go to** L3; **go to** L4;

L2: jm := nop-1; js := 1; **go to** term;

L3: jm := 2; js := 1; **go to** term;

**comment** Locate argument;

L4: **for** ia := 2 **step** 1 **until** nop **do begin**  
**if** xa [ia] > xarg **then go to** L7; jm := ia **end**;

**comment** Before loop is complete xarg  $\leq$  xa [ia];

L5: ca := a; cb := b; cc := c; js := 3; jm := jm+1; **go to** term;

L6: a := (ca+a)/2; b := (cb+b)/2; c := (cc+c)/2; **go to** L9;

L7: js := 2; **go to** term;

L8: **go to** beta [js];

L9: **go to** gamma [jt];

**comment** Interpolation, jt = 1;

L10: da := a  $\times$  xarg  $\uparrow$  2 + b  $\times$  xarg + c; **go to** exit1;

**comment** Differentiation, jt = 2;

L11: dif := 2  $\times$  xarg + b; **go to** exit2;

**comment** Integration, jt = 3;

L12: sum := 0; syl := xlo; jul := nop - 1;  
ib := 2;

L16: **for** jm := ib **step** 1 **until** jul **do begin**;

**comment** Lagrange formulae;

term1 := ya [jm - 1] / ((xa [jm - 1] - xa [jm])  $\times$  (xa [jm - 1] - xa [jm + 1]));

term2 := ya [jm] / ((xa [jm] - xa [jm - 1])  $\times$  (xa [jm] - xa [jm + 1]));

term3 := ya [jm + 1] / ((xa [jm + 1] - xa [jm - 1])  $\times$  (xa [jm + 1] - xa [jm]));

a := term1 + term2 + term3;

b := -(xa [jm] + xa [jm + 1])  $\times$  term1 - (xa [jm - 1] + xa [jm + 1])  $\times$  term2 - (xa [jm - 1] + xa [jm])  $\times$  term3;

c := xa [jm]  $\times$  xa [jm + 1]  $\times$  term1 + xa [jm - 1]  $\times$  xa [jm + 1]  $\times$  term2 + xa [jm - 1]  $\times$  xa [jm]  $\times$  term3; **go to** delta [jt];

L13: **if** jm  $\neq$  2 **then go to** L14;

ca := a; cb := b; cc := c; **go to** L15;

L14: ca := (a + ca)/2; cb := (b + cb)/2; cc := (c + cc)/2;

L15: syu := xa [jm];  
sum := sum + ca  $\times$  (syu  $\uparrow$  3 - syl  $\uparrow$  3)/3 + cb  $\times$  (syu  $\uparrow$  2 - syl  $\uparrow$  2)/2 + cc  $\times$  (syu - syl);  
ca := a; cb := b; cc := c; syl := syu **end**;

**comment** End of loop on [jm] index;

sum := sum + ca  $\times$  (xup  $\uparrow$  3 - syl  $\uparrow$  3)/3 + cb  $\times$  (xup  $\uparrow$  2 - syl  $\uparrow$  2)/2 + cc  $\times$  (xup - syl); **go to** exit3;

**term:** ib := jm; jul := ib; **go to** L16;

**comment** The results for interpolation, differentiation, and integration are da, dif, and sum respectively;

exit1: AVINT := da; **go to** exit;

exit2: AVINT := dif; **go to** exit;

exit3: AVINT := sum;

**exit:** **end**

ALGORITHM 78  
RATIONAL ROOTS OF POLYNOMIALS WITH IN-  
TEGER COEFFICIENTS

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**comment** This ALGOL procedure, named *ratfact*, for finding rational roots of polynomials with integer coefficients is a pedagogical example illustrating the use of the **for** statement described in section 4.6.3. Also, an extension suggested by J. Peck of the well-known polynomial evaluation by nesting, i.e. Horner's method, is used. The polynomial  $f(x) = a_0 + a_1x + \dots + a_nx^n$  with integer coefficients and with  $a_0a_n \neq 0$  has a lowest term rational root  $p/q$  if and only if  $a_0q^n + a_1q^{n-1}p + \dots + a_{n-1}q p^{n-1} + a_np^n = 0$ , also  $q$  must be a factor of  $a_n$  and  $p$  a factor of  $a_0$ . Procedure *RATFACT* outputs the nonzero rational roots  $p/q$  by execution of the procedure whose formal name is *print*. The output procedure uses the string whose formal name is *format* for control of the output format;

```
procedure ratfact (a, n, print, format);
integer array a[0:n]; integer n; procedure print; string
format;
begin integer i, p, q, r, t, f, g;
p loop: for p := 1 step 1 until abs (a[0]) do
begin comment if p is not a factor of a [0] or q is not a factor
of a[n] then skip to the end of the loop for advance in the
respective for list;
if a[0]  $\neq$  (a[0] $\div$ p) $\times$ p then go to 1
else q loop: for q := 1 step 1 until abs (a[n]) do
begin if a[n]  $\neq$  (a[n] $\div$ q) $\times$ q then go to 2
else
begin comment root test and print;
comment start polynomial evaluation;
f := g := a[0]; t := p;
for i := 1 step 1 until n do
begin r := a[i] $\times$ t;
f := f $\times$ q+r;
g := -g $\times$ q+r;
t := t $\times$ p;
end polynomial evaluation;
comment computing r saves one subscript
evaluation;
if f=0 then print (format, p, q);
if g=0 then print (format,-p, q);
comment print is the formal name of the procedure
to be used to output the variables in the format
specified by the string whose formal name is format;
end root test and print;
2: end q loop;
1: end p loop;
end ratfact, without overflow test.
```

ALGORITHM 79  
DIFFERENCE EXPRESSION COEFFICIENTS  
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```
procedure dicol (k, n, xp, xtab, coef);
value k, n; integer k, n; real xp;
array xtab, coef;
comment dicol produces the coefficients for the n ordinates
(corresponding to the abscissae, xtab) in the n-point finite
difference expression for the k-th derivative evaluated at xp.
The method used is to determine the analytic expression for
the k-th derivative of each coefficient in the n-point Lagrangian
interpolation formula and evaluate it at xp. Note that k=0
will produce the Lagrangian interpolation coefficients them-
selves;
begin integer array xuse [1 : n-1]; real factk, sum, denom,
part;
integer i, terms, j, m, high;
factk := 1.0; for i := 2 step 1 until k do factk := i $\times$ factk;
terms := n-k-1; if terms<0 then go to Z;
for j := 1 step 1 until n do
loop: begin sum := 0; denom := 1.0; part := 1.0;
for i := 1 step 1 until n do
if i  $\neq$  j then denom := denom $\times$ (xtab [j] - xtab [i]);
if terms = 0 then go to Y;
m := 1; high := 1;
A: if (high = j) $\sqrt$ (xtab [high] = xp) then
A1: begin high := high + 1; go to A end A1;
if high > n then A2: begin m := m-1; if m>0
then
A3: begin high := xuse [m]+1; go to A end A3;
go to X end A2;
xuse [m] := high; m := m+1;
if m $\leq$ terms then begin high := high + 1; go to
A end;
for i := 1 step 1 until terms do
part := part $\times$ (xp - xtab [xuse [i]]);
sum := sum + part; m := terms; part := 1.0;
high := xuse [terms] + 1; go to A;
Y: sum := 1.0;
X: coef [j] := sum  $\times$  factk/denom end loop;
go to EXIT;
Z: for i := 1 step 1 until n do coef [i] := 0;
EXIT: end dicol
```



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