Algorithms

H. J. WEGSTEIN, Editor

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ALGORITHM 80
RECIPROCAL GAMMA FUNCTION OF REAL ARGUMENT

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```
real procedure RGR(x); real x; real procedure RGAM;
comment Procedure RGAM computes the real reciprocal
  Gamma function of real x for −1 < x < 1, utilizing Horner's
  method for polynomial evaluation of the approximation poly-
  nomial. RGR extends the range of RGAM by use of the formulae
  (1) 1/Gamma(x−1) = (x−1)/Gamma(x) for x < −1,
  (2) 1/Gamma(x+1) = 1/x×Gamma(x) for x<1.;
  begin  real y;
         if x = 0 then begin RGR := 0;  go to EXIT end
         if x = 1 then begin RGR := 1;  go to EXIT end
         if x < 1 then go to BB;
         y := 1;
AA:      x := x − 1;  y := y × x;  if x > 1 then go to AA;
         if x = 1 then begin RGR := 1/y;  go to EXIT end
         RGR := RGAM(x)/y;  go to EXIT;
BB:      if x = −1 then begin RGR := 0;  go to EXIT end
         if x > −1 then begin RGR := RGAM(x);
           go to EXIT end
         y := x;
CC:      x := x + 1;  if x < −1 then begin y := y × x;
           go to CC end
         RGR := RGAM(x) × y;
EXIT:    end RGR;
```

```
real procedure RGAM(x);  real x;  integer i;
  real array B[0:13];
comment The algorithm for this routine was adapted from
  "University of Illinois Digital Computer, Auxiliary Library
  Routine B-17-328", by John Ehrman. Reference may also be
  made to Algorithm 34, dated February, 1961. Approximation
  accuracy is ±2^-35.;
begin real z;
  B[ 0] :=  1.00000 00000 00;  B[ 1] := −.42278 43350 92;
  B[ 2] := −.23309 37363 65;  B[ 3] := +.19109 11011 62;
  B[ 4] := −.02455 24908 87;  B[ 5] := −.01764 52421 18;
  B[ 6] := +.00802 32781 13;  B[ 7] := −.00080 43413 35;
  B[ 8] := −.00036 08514 96;  B[ 9] := +.00014 56243 24;
  B[10] := −.00001 75279 17;  B[11] := −.00000 26257 21;
  B[12] := +.00000 13285 54;  B[13] := −.00000 01812 20;
  z: = B[13];
  for i := 12 step −1 until 0 do z := z × x + B[i];
  RGAM := z × x × (x + 1)
end RGAM;
```

ALGORITHM 81
ECONOMISING A SEQUENCE 1

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```
procedure ECONOMISER 1 (desired property, costs, n, C);
  array costs;  integer n;
    Boolean procedure desired property;
  Boolean array C;
begin comment Given a finite, monotonely increasing
  sequence of positive numbers, looked upon as prices, ECONO-
  MISER 1 selects the cheapest subsequence with a given prop-
  erty. The formal parameters are: Desired property, a function
  designator to answer the question: Does the subsequence held
  in array C possess the required property? n is (number of ele-
  ments in the sequence) + 1. Costs is an array of size [1:n].
  Costs[1] to costs[n−1] hold the numbers of the sequence and
  costs[n] is any arbitrary number greater than the sum of all
  other elements of costs. C is an array of the same size and indi-
  cates a subsequence by the rule: C[i] ≡ element i of the original
  sequence is in the subsequence. At exit from ECONOMISER 1,
  C indicates the cheapest subsequence. It is supposed that the
  original sequence has the desired property.;
  integer d, j, k, ℓ;  real i;
  for j := 1 step 1 until n do C[j] := j = 1;  d := 0;
  reenter:  d := d+1;
  INSIDE:  begin own real array prices [1:d];
                own Boolean array alternatives[1:d, 1:n];
                procedure ENTER SUCCESSORS;
                begin k := n−1;
                  A:  if ¬ C[k] then
                      begin k := k−1;  go to A end;  i := 0;
                      for j := 1 step 1 until n do
```

```
                    begin alternatives[ℓ,j]
                      := j ≠ k ∧ j ≠ k−1 ≡ C[j];
                      if alternatives[ℓ,j] then
                      i := i + costs[j]
                    end;
                B:  k := k−1;
                    go to if k = 0 then find cheapest
                      else if C[k] then (if k=1 then
                        find cheapest else B)
                      else if k=1 then E
                        else if C[k−1] then D
                      else find cheapest;
                D:  C[k−1] := false;
                E:  C[k] := true;  go to reenter
              end of ENTER SUCCESSORS;
              i := 0;  for j := 1 step 1 until n do
              begin alternatives[d,j] := C[j];  if C[j] then
                i := i + costs[j]
              end;  prices[d] := i;
            find cheapest:  i := 0;  for j := 1 step 1 until d do
              begin if prices[j] < i then
                begin ℓ := j;  i := prices[ℓ] end
              end;
              for j := 1 step 1 until n do
              C[j] := alternatives[ℓ,j];
              if ¬ desired property then
                ENTER SUCCESSORS
            end of INSIDE;
end of ECONOMISER 1;
```

ALGORITHM 82
ECONOMISING A SEQUENCE 2

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```
procedure  ECONOMISER 2 (desired property, costs, n, C, r,
  Reject list);  Boolean procedure desired property;
  integer n, r;  array costs;  Boolean array Reject list;
begin comment  In some applications of ECONOMISER 1, it
  is simple to establish that some subsequences are redundant in
  the sense that any sequence containing them is certainly not
  the cheapest subsequence with the desired property. For such
  applications ECONOMISER 2 avoids all unnecessary calls of
  desired property. The new formal parameters are:  r a variable
  whose value is initially 0 and is increased by 1 every time that
  desired property discovers a new redundant subsequence.
  Reject list an array of size [1:r,1:n]. Reject list [a,b] carries the
  answer to: Is element b of the original sequence in the a^th
  redundant subsequence found by desired property?;
  real i;  integer d, j, k, ℓ;  Boolean gapfilled, first time;
  procedure INSIDE (entrymaker);  Boolean entrymaker:
  begin own real array prices[1:d];
        own Boolean array alternatives[1:d,1:n];
        procedure ENTER SUCCESSORS;
        begin integer c;  Boolean array ssq[1:n];
            for j := 1 step 1 until n do ssq[j] := C[j];
            c := n−1;
        A:  if ¬ ssq[c] then begin c := c−1;  go to A end:
            C[c] := false;  C[c+1] := true;
            INSIDE (true);
            gapfilled := true;
        B:  c := c−1;
            go to if c=0 then F else if ssq[c] then
              (if c=1 then F else B) else if c=1 then
              E else if ssq[c−1] then D else F;
        D:  ssq[c−1] := false;
        E:  for j := 1 step 1 until n do C[j] := ssq[j] ≡ j≠c;
            INSIDE (true);
        F:  end of ENTER SUCCESSORS;
        if entrymaker then
        begin for j := 1 step 1 until r do
            begin for k := 1 step 1 until n do
              begin if ¬ C[k] ∧ Reject list[j,k] then
                go to G end;
              ENTER SUCCESSORS;  go to H;
        G:  end;
            i := 0;  if gapfilled then d := d+1;
            for j := 1 step 1 until n do
            begin  alternatives[if gapfilled  then
              d else ℓ, j] := C[j];
              if C[j] then i := i + costs[j]
            end;  prices[if gapfilled then d else ℓ] := i
        end;  if first time ∨ ¬ entrymaker then
        begin i := 0;  gapfilled := first time := false;
          for j := 1 step 1 until d do
          begin if prices[j] < i then
            begin ℓ := j;  i := prices[ℓ] end
          end;
            for j := 1 step 1 until n do
              C[j] := alternatives[ℓ,j];
          if desired property then go to found;
          ENTER SUCCESSORS;  go to reenter
          end;
  H:  end of INSIDE;
  for j := 1 step 1 until n do C[j] := j=1;
  d := 0;  first time := gapfilled := true;
reenter:  INSIDE (first time);
found:
end of ECONOMISER 2;
```

ALGORITHM 83
OPTIMAL CLASSIFICATION OF OBJECTS

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```
procedure  OPTIMUM COVERING FINDER (Pattern, popu-
  lation, set number, set prices, chosen sets, bounds, overflow);
  Boolean array Pattern, chosen sets;  integer population,
  set number, bounds;  array set prices;  label overflow;
begin comment  The number of objects in some given set is
  given by population. The procedure is given a classification of
  these objects by a collection of overlapping subsets. A cost
  is assigned to each subset. Then OPTIMUM COVERING
  FINDER selects the cheapest subcollection such that every
  object is contained in at least one of the subsets of the sub-
  collection.  set prices[i] carries the cost of subset i.  Pattern
  is an array of size [1:set number,1:population] such that Pat-
  tern[a,b] ≡ does subset a include object b.  chosen sets[i] finally
  carries the answer to the question: Is set i in the cheapest
  subcollection? The programmer must restrict the amount of
  space available to the procedure by setting bounds. From ex-
  perience bounds = set number ↑ 2 suffices to avoid most alarm
  exits to overflow.;
  Boolean array C[1:population], D[1:bounds, 1:population],
    R, S[1:bounds,1:set number];
  integer a, b, d, r, s;
  Boolean procedure HAVE WE A COVERING;
  begin procedure ADD to (Q,q,f);  integer q;
          real f;  Boolean array Q;
      begin if q=bounds then go to overflow else q := q+1;
        for a := 1 step 1 until set number do Q[q,a] := f
      end;  for a := 1 step 1 until population do
```

```
            C[a] := false;
    for a := 1 step 1 until set number do
    begin if chosen sets[a] then
      for b := 1 step 1 until population do
      C[b] := C[b] ∨ Pattern[a,b]
    end;  for a := 1 step 1 until population do
    begin if ¬ C[a] then go to E end;
    go to found;
 E: for d := 1 step 1 until s do
    begin for b := 1 step 1 until population do
      begin if C[b] ∧ ¬ D[d,b] then go to try another end;
      ADD to (R, r, chosen sets[a]);
      for b := 1 step 1 until set number do
      begin if chosen sets[b] ∧ ¬ S[d,b] then
        ADD to (R, r, S[d,a] ∨ a=b)
      end;  go to F;
    try another:
    end of for statement labelled E;
    ADD to (S, s, chosen sets[a]);
    for a := 1 step 1 until population do D[s,a] := C[a];
 F: HAVE WE A COVERING := false
 end;  r := s := 0;
 ECONOMISER 2 (HAVE WE A COVERING, set prices,
   set number, r, R, chosen sets);
found:  end
```

CERTIFICATION OF ALGORITHM 60
ROMBERG INTEGRATION (F. L. Bauer, *Comm. ACM*, June, 1961)

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This procedure was translated to the ACT III compiler language for the Royal Precision LGP-30 computer. This system provides 7+ significant decimal digits. The program was used to integrate x^n between the limits 0.01 and 1.1, and between the limits 1.1 and 0.01. The results in Table I were obtained. The pole at 0 for negative n affords a test of the reliability of the method when the higher derivatives of the integrand are large. The agreement between integrations in the forward and backward directions is an indication of the effects of round-off error.

It is apparent that the procedure gives results well within the noise level for the positive powers, and that even the effect of a closely adjacent singularity for the negative powers can be overcome.

TABLE I. Integration of $\int_{0.01}^{1.1} x^n dx$ and $\int_{1.1}^{0.01} x^n dx$

n	0	+12	+12	−1
True Value	1.0900000	.26555932	−.26555932	4.7004831
Order 1	1.0899997	.57076812	−.57076842	19.641113
Order 2	1.0899997	.30614608	−.30614626	10.656923
Order 5	1.0899991	.26555693	−.26555818	4.9017590
Order 10				4.7002345

n	−1	−5	−5
True Value	−4.7004831	$.25000000\times10^8$	-18.166667×10^8
Order 1	−19.641125	18.166655×10^8	$-.25000000\times10^8$
Order 2	−10.656929	8.4777719×10^8	-8.4777766×10^8
Order 5	−4.9017805	1.0408634×10^8	-1.0408640×10^8
Order 10	−4.7004402	$.25000715\times10^8$	$-.25000727\times10^8$
Order 12		$.24999291\times10^8$	$-.25001311\times10^8$

The flexibility of the algorithm would be improved by adding to the formal parameters a procedure, check, to decide if sufficient accuracy had been obtained without carrying through the entire iteration. A possible form for this procedure would be:

```
procedure check (t1, t2, f, exit);
  real t1, t2;
  label exit;
  integer f;
begin if abs ((t2 − t1) × f) / t1 < tolerance ∧ f > minimum order
  then go to exit end.
```

The global variables tolerance, which is the maximum relative difference between approximations of increasing order, and the minimum acceptable order should be selected by the programmer for the exigencies of the problem. A check of this sort is clearly not as sound as an a priori estimate of the necessary order, but is frequently an acceptable expedient.

The Romberg quadrature algorithm is analyzed in the following references:

Romberg, W. Vereinfachte numerische Integration. *Det Kongelinge Norske Videnskaber Selskab Forhandlinger 28*, (1955), 30-36.

Stiefel, E., and Rutishauser, H. Remarques concernant l'integration numerique. *Comptes Rendus Acad. Scil (Paris) 252*, (1961), 1899-1900.

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CERTIFICATION OF ALGORITHM 78
RATFACT (C. Perry, *Comm. ACM 5*, Feb. 1962)

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RATFACT was copied in the Navy Electronics Laboratory International Algol Compiler, Neliac, and tested on the Univac M-490 Countess and the CDC 1604. Polynomials of order 2 through 6 were tested. No corrections were found necessary. It was noted that a polynomial whose coefficients included a common factor would produce superfluous values of p/q, in which this fraction was indeed a root, but one in which p and q contained a common factor.