smallest of these integers is +1. ∘ designates the empty relation, so that $x \circ y$ is true for arbitrary x, y. If M is such that no f and g exist which satisfy all n^2 relations, then control is transferred to the label parameter *fail*. This procedure has been used to determine the precedence functions of symbols in a given precedence grammar (see [Floyd, R. Syntactic analysis and operator precedence. *J.ACM 10* (1963), 316–333]);

```
begin integer i, j, k, k1, fmin, gmin;
  procedure fixrow (i, l, x);  value i, l, x;  integer i, l, x;
  begin integer j; f[i] := g[l] :+ x;
    if k = k1 then
    begin if M[i, k] = ls ∧ f[i] ≥ g[k] then go to fail else
      if M[i, k] = eq ∧ f[i] ≠ g[k] then go to fail
    end;
    for j := k1 step −1 until 1 do
    if M[i, j] = ls ∧ f[i] ≥ g[j] then fixcol (i, j, 1) else
    if M[i, j] = eq ∧ f[i] ≠ g[j] then fixcol (i, j, 0)
  end fixrow;
  procedure fixcol (l, j, x); value l, j, x; integer l, j, x;
  begin integer i; g[j] := f[l] + x;
    if k ≠ k1 then
    begin if M[k,j] = gr ∧ f[k] ≤ g[j] then go to fail else
      if M[k,j] = eq ∧ f[k] ≠ g[j] then go to fail
    end;
    for i := k step −1 until 1 do
    if M[i, j] = gr ∧ f[i] ≤ g[j] then fixrow (i, j, 1) else
    if M[i, j] = eq ∧ f[i] ≠ g[j] then fixrow (i, j, 0)
  end fixcol;
  k1 := 0;
  for k := 1 step 1 until n do
  begin fmin := 1;
    for j := 1 step 1 until k1 do
      if M[k, j] = gr ∧ fmin ≤ g[j] then fmin := g[j]+1 else
      if M[k, j] = eq ∧ fmin < g[j] then fmin := g[j];
    f[k] := fmin;
    for j := k1 step −1 until 1 do
      if M[k, j] = ls ∧ fmin ≥ g[j] then fixcol (k, j, 1) else
      if M[k, j] = eq ∧ fmin > g[j] then fixcol (k, j, 0);
    k1 := k1+1; gmin := 1;
    for i := 1 step 1 until k do
      if M[i, k] = ls ∧ f[i] ≥ gmin then gmin := f[i]+1 else
      if M[i, k] = eq ∧ f[i] > gmin then gmin := f[i];
    g[k] := gmin;
    for i := k step −1 until 1 do
      if M[i, k] = gr ∧ f[i] ≤ gmin then fixrow (i, k, 1) else
      if M[i, k] = eq ∧ f[i] < gmin then fixrow (i, k, 0)
  end k
end Precedence
```

ALGORITHM 266
PSEUDO-RANDOM NUMBERS [G5]

M. C. Pike and I. D. Hill
(Recd. 15 Feb. 1965 and 6 July 1965)
Medical Research Council, London, England

```
real procedure random (a, b, y);
  real a, b;  integer y;
comment random generates a pseudo-random number in the
  open interval (a, b) where a < b. The procedure assumes that
  integer arithmetic up to 3125 × 67108863 = 209715196875 is
  available. The actual parameter corresponding to y must be an
  integer identifier, and at the first call of the procedure its value
  must be an odd integer within the limits 1 to 67108863 inclusive.
  If a correct sequence is to be generated, the value of this inte-
  ger identifier must not be changed between successive calls of
  the procedure;
begin
  y := 3125 × y;  y := y − (y÷67108864) × 67108864;
  random := y/67108864.0 × (b−a) + a
end random
```

Coveyou [2] showed that for multiplicative congruential methods of generating pseudorandom numbers, the correlation between successive numbers will be approximately the reciprocal of the multiplying factor. Greenberger [3] showed further that the factor should be considerably less than the square root of the modulus.

(*continued on next page*)

Revised Algorithms Policy • May, 1964

A contribution to the Algorithms department must be in the form of an algorithm, a certification, or a remark. Contributions should be sent in duplicate to the editor, typewritten double-spaced in capital and lower-case letters. Authors should carefully follow the style of this department, with especial attention to indentation and completeness of references. Material to appear in **boldface** type should be underlined in black. Blue underlining may be used to indicate *italic* type, but this is usually best left to the Editor.

An algorithm must be written in the Algol 60 Reference Language [*Comm. ACM 6* (Jan. 1963), 1–17], and normally consists of a commented procedure declaration. Each algorithm must be accompanied by a complete driver program in Algol 60 which generates test data, calls the procedure, and outputs test answers. Moreover, selected previously obtained test answers should be given in comments in either the driver program or the algorithm. The driver program may be published with the algorithm if it would be of major assistance to a user.

Input and output should be achieved by procedure statements, using one of the following five procedures (whose body is not specified in Algol): [see "Report on Input-Output Procedures for ALGOL 60," *Comm, ACM 7* (Oct. 1964), 628–629].

```
procedure inreal (channel, destination); value channel; integer channel;
  real destination; comment the number read from channel channel is as-
  signed to the variable destination; . . . ;
procedure outreal (channel, source); value channel, source; integer channel;
  real source; comment the value of expression source is output to channel
  channel; . . . ;
procedure ininteger (channel, destination);
  value channel; integer channel, destination; . . . ;
procedure outinteger (channel, source);
  value channel, source; integer channel, source; . . . ;
procedure outstring (channel, string); value channel; integer channel;
  string string; . . . ;
```

If only one channel is used by the program, it should be designated by 1. Examples:

```
outstring (1, 'x ='); outreal (1, x);
for i := 1 step 1 until n do outreal (1, A[i]);
ininteger (1, digit [17]);
```

It is intended that each published algorithm be a well-organized, clearly commented, syntactically correct, and a substantial contribution to the Algol literature. All contributions will be refereed both by human beings and by an Algol compiler. Authors should give great attention to the correctness of their programs, since referees cannot be expected to debug them. Because Algol compilers are often incomplete, authors are encouraged to indicate in comments whether their algorithms are written in a recognized subset of Algol 60 [see "Report on SUBSET ALGOL 60 (IFIP)," *Comm. ACM 7* (Oct, 1964), 626–627].

Certifications and remarks should add new information to that already published. Readers are especially encouraged to test and certify previously uncertified algorithms. Rewritten versions of previously published algorithms will be refereed as new contributions, and should not be imbedded in certifications or remarks.

Galley proofs will be sent to the authors; obviously rapid and careful proofreading is of paramount importance.

Although each algorithm has been tested by its author, no liability is assumed by the contributor, the editor, or the Association for Computing Machinery in connection therewith.

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The method of Algorithm 133 [1] satisfies Greenberger's condition, but since the reciprocal of its multiplying factor is as high as 0.2, Coveyou's result shows that it is very unsatisfactory for purposes requiring statistically independent consecutive random numbers.

Algorithms 133 and 266 have both been tested by computing a number of sets of 2000 successive random integers between 0 and 9, dividing each set into 400 groups of 5, and performing the poker test [4]. The results were classified in the following seven categories:

- *(i)* all different
- *(ii)* 1 pair
- *(iii)* 2 pairs
- *(iv)* 3 of a kind
- *(v)* 3 of a kind and 1 pair
- *(vi)* 4 of a kind
- *(vii)* 5 of a kind.

The following tables resulted:

ALGORITHM 133

Run	Starting Value	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
1	13421773	114	193	42	37	7	7	0
2	22369621	111	181	46	40	14	8	0
3	33554433	130	178	48	28	7	6	3
4	6871947673	118	179	51	35	10	5	2
5	11453246123	128	189	44	28	6	4	1
6	17179869185	135	155	45	52	6	5	2
Expected for each Run		120.96	201.60	43.20	28.80	3.60	1.80	0.04
Total for 6 Runs		736	1075	276	220	50	35	8
Expected for Total		725.76	1209.60	259.20	172.80	21.60	10.80	0.24

ALGORITHM 266

Run	Starting Value	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
1	13421773	132	191	35	38	2	2	0
2	22369621	140	187	45	27	0	1	0
3	33554433	129	198	44	25	4	0	0
4	8426219	107	202	50	37	2	2	0
5	42758321	101	207	60	25	5	2	0
6	56237485	118	203	42	34	1	2	0
7	62104023	119	206	41	27	6	1	0
Expected for each Run		120.96	201.60	43.20	28.80	3.60	1.80	0.04
Total for 7 Runs		846	1394	317	213	20	10	0
Expected for Total		846.72	1411.20	302.40	201.60	25.20	12.60	0.28

Combining categories *(vi)* and *(vii)* in each case, the observed totals give χ^2 values (on 5 degrees of freedom) of 159.0 for Algorithm 133, and of 3.28 for Algorithm 266.

References:

1. Behrenz, P. G. Algorithm 133, Random. *Comm. ACM 5* (Nov. 1962), 553.
2. Coveyou, R. R. Serial correlation in the generation of pseudo-random numbers. *J. ACM 7*(1960), 72–74.
3. Greenberger, M. An a priori determination of serial correlation in computer generated random numbers. *Math. Comput. 15*(1961), 383–389. Correction in *Math. Comput.16*(1962), 126.
4. Kendall, M. G., and Babington-Smith, B. Randomness and random sampling numbers. *J. Royal Statist. Soc. 101* (1938), 147–166.

ALGORITHM 267
RANDOM NORMAL DEVIATE [G5]

M. C. Pike (Recd. 3 May 1965 and 6 July 1965)
Medical Research Council, London, England


```
procedure RND(x1, x2, Random);
  real procedure Random;   real x1, x2;
comment RND uses two calls of the real procedure Random
  which is any pseudo-random number generator which will
  produce at each call a random number lying strictly between 0
  and 1. A suitable procedure is given by Algorithm 266, Pseudo-
  Random Numbers [Comm. ACM 8(Oct. 1965), 605] if one chooses
  a = 0, b = 1 and initializes y to some large odd number, such as
  13421773. RND produces two independent random variables x1
  and x2 each from the normal distribution with mean 0 and
  variance 1. The method used is given by Box, G.E.P., and
  Muller, M.E., A note on the generation of random normal
  deviates. [Ann. Math. Stat. 29 (1958), 610–611];
begin real t;
  x1 := sqrt(−2.0 × ln(Random));
  t := 6.2831853072 × Random;
  comment 6.2831853072 = 2 × pi;
  x2 := x1 × sin(t);   x1 := x1 × cos(t)
end RND
```

Algorithm 121, NormDev [*Comm. ACM 5* (Sept. 1962), 482; *8* (Sept. 1965), 556] also produces random normal deviates and Algorithm 200, NORMAL RANDOM [*Comm. ACM 6* (Aug. 1963), 444; *8* (Sept. 1965), 556] produces random deviates with an approximate normal distribution, but the procedure *RND* seems preferable to both of them.

We may compare *NORMAL RANDOM* to *RND* (which is exact) by noting that at recommended minimum *n NORMAL RANDOM* requires 10 calls of *Random* while *RND* gets two independent normal deviates from 2 calls of *Random* and one call each of *sqrt*, *ln*, *sin* and *cos*. Under the stated test conditions a single call of *NORMAL RANDOM* (with $n = 10$) took 20 percent more computing time than a single call of *RND* when the real procedure *Random* was given by Algorithm 266.

To compare *NormDev* to *RND* in the same way, we have first to calculate the expected number of calls of *ln*, *sqrt*, *exp* and *Random* for each call of *NormDev*. This may be done by noting that there is (1) an initial single call of *Random*, then (2) with probability 0.68 a random normal deviate restricted to (0, 1) has to be found and this requires on average 1.36 calls of *Random* and 1.18 calls of *exp*, and (3) with probability 0.32 a random normal deviate restricted to $(1, \infty)$ has to be found and this requires on average 2.04 calls of *Random* and 1.52 calls of each of *ln* and *sqrt*. *NormDev* thus requires on average 2.58 calls of *Random*, 0.80 calls of *exp*, 0.49 calls of *ln* and 0.49 calls of *sqrt*. (Note: *NormDev* requires one further call of *Random* if a signed normal deviate is required.) Under the stated test conditions a single call of *NormDev* took virtually the same amount of computing time as a single call of *RND* when the real procedure *Random* was as above.

(Note: In testing *NormDev* the procedure was speeded up by replacing *A* by 0.6826894 wherever it occurred and removing it from the parameter list. In testing *NORMAL RANDOM Mean, Sigma, n* were replaced by 0, 1.0 and 10 respectively and removed from the parameter list.)