```
  hz := hf := 0;
  for i := 1 step 1 until n do
  begin
    x[i] := x[i] − dx[i];  hz := hz + abs(x[i]);
    hf := hf + abs(dx[i])
  end;
  if hf ≥ eps2 × hz then go to ITERATION;
ENDE: FUNCTION(x, f);  S := 0;  itmax := z;
  for i := 1 step 1 until m do S := S + f[i] × f[i]
end TAYLOR
```


REFERENCE:

[1] BRAESS, D. Über Dämpfung bei Minimalisierungsverfahren. *Computing 1* (1966), 264–272.

ALGORITHM 316
SOLUTION OF SIMULTANEOUS NON-LINEAR EQUATIONS [C5]

K. M. BROWN (Recd. 27 Oct. 1966, 31 Mar. 1967, 17 July 1967, and 26 July 1967)

Department of Computer Science, Cornell University, Ithaca, New York


```
procedure nonlinearsystem (n, maxit, numsig, singular, x);
  value n, numsig;  integer n, maxit, numsig, singular;  array x;
comment This procedure solves a system of n simultaneous
  nonlinear equations. The method is roughly quadratically con-
  vergent and requires only ((n²/2)+(3n/2)) function evaluations
  per iterative step as compared with (n²+n) evaluations for
  Newton's Method. This results in a savings of computational
  effort for sufficiently complicated functions. A detailed de-
  scription of the general method and proof of convergence are
  included in [1]. Basically the technique consists in expanding
  the first equation in a Taylor series about the starting guess,
  retaining only linear terms, equating to zero and solving for
  one variable, say x_k , as a linear combination of the remaining
  n − 1 variables. In the second equation, x_k is eliminated by
  replacing it with its linear representation found above, and
  again the process of expanding through linear terms, equating
  to zero and solving for one variable in terms of the now remain-
  ing n − 2 variables is performed. One continues in this fashion,
  eliminating one variable per equation, until for the nth equa-
  tion, we are left with one equation in one unknown. A single
  Newton step is now performed, followed by back-substitution
  in the triangularized linear system generated for the x_i's. A
  pivoting effect is achieved by choosing for elimination at any
  step that variable having a partial derivative of largest absolute
  value. The pivoting is done without physical interchange of
  rows or columns.
    The vector of initial guesses x, the number of significant digits
  desired numsig, the maximum number of iterations to be used,
  maxit, and the number of equations n, should be set up prior to
  the procedure call which activates nonlinearsystem. After execu-
  tion of the procedure, the vector x is the solution of the system
  (or best approximation thereto), maxit is now the number of
  iterations used and singular = 0 is an indication that a Jaco-
  bian-related matrix was singular—indicative of the process
  "blowing-up," whereas singular = 1 is an indication that no
  such difficulty occurred. Storage space may be saved by imple-
  menting the algorithm in a way which takes advantage of the
  fact that the strict lower triangle of the array pointer and the
  same number of positions in the array coe are not used;
begin integer converge, m, j, k, i, jsub, itemp, kmax, kplus, tally;
  real f, hold, h, fplus, dermax, test, factor, relconvg;
  integer array pointer[1:n, 1:n], isub[1:n−1];
  array temp, part[1:n], coe[1:n, 1:n+1];
  procedure backsubstitution (k, n, x, isub, coe, pointer);
    value k, n;
    integer k, n;  integer array isub, pointer;  array x, coe;
  comment This procedure back-solves a triangular linear
    system for improved x[i] values in terms of old ones;
  begin integer km, kmax, jsub;
    for km := k step −1 until 2 do
    begin kmax := isub[km−1];  x[kmax] := 0;
      for j := km step 1 until n do
      begin jsub := pointer[km, j];
        x[kmax] := x[kmax] + coe[km−1, jsub] × x[jsub]
      end;
      x[kmax] := x[kmax] + coe[km−1, n+1]
    end;
  end backsubstitution;
  procedure evaluatekthfunction (x, y, k);
    integer k;  real y;  array x;
  begin comment the body of this procedure must be provided
    by the user. One call of the procedure should cause the value
    of the kth function at the current value of the vector x to be
    placed in y;
  end evaluatekthfunction;
  converge := 1;  singular := 1;  relconvg := 10↑(-numsig);
  for m := 1 step 1 until maxit do
  begin
    comment An intermediate output statement may be in-
      serted at this point in the procedure to print the successive
      approximation vectors x generated by each complete itera-
      tive step;
    for j := 1 step 1 until n do pointer [1, j] := j;
    for k := 1 step 1 until n do
    begin if k > 1 then backsubstitution (k, n, x, isub, coe, pointer);
      evaluatekthfunction (x, f, k);  factor := .001;
AAA:      tally := 0;  for i := k step 1 until n do
      begin itemp := pointer[k, i];  hold := x[itemp];
        h := factor × hold;  if h = 0 then h := .001;
        x[itemp] := hold + h;
        if k > 1 then backsubstitution (k, n, x, isub, coe, pointer);
        evaluatekthfunction (x, fplus, k);
        part[itemp] := (fplus−f)/h;
        x[itemp] := hold;  if (abs(part[itemp])=0)∨
        (abs(f/part[itemp]) > 1.0₁₀20) then tally := tally + 1;
      end;
      if tally ≦ n − k then go to AA;  factor := factor × 10.0;
      if factor > .5 then go to SING;  go to AAA;
AA:   if k < n then go to A;  if abs (part[itemp]) = 0
      then go to SING;
      coe[k, n+1] := 0;  kmax := itemp;  go to ENDK;
A:    kmax := pointer[k, k];  dermax := abs(part[kmax]);
      kplus := k + 1;
      for i := kplus step 1 until n do
      begin jsub := pointer[k, i];  test := abs(part[jsub]);
        if test < dermax then go to B;  dermax := test;
        pointer [kplus, i] := kmax;  kmax := jsub;
        go to ENDI;
B:      pointer [kplus, i] := jsub;
ENDI:
      end;
      if abs(part[kmax]) = 0 then go to SING;  isub[k] := kmax;
      coe[k, n+1] := 0;
      for j := kplus step 1 until n do
      begin jsub := pointer[kplus, j];
        coe[k, jsub] := −part[jsub]/part[kmax];
        coe[k, n+1] := coe[k, n+1] + part[jsub] × x[jsub]
      end;
ENDK:
      coe[k, n+1] := (coe[k, n+1]−f)/ part[kmax] + x[kmax]
    end k;
    x[kmax] := coe[n, n+1];
    if n > 1 then backsubstitution (n, n, x, isub, coe, pointer);
    if m = 1 then go to D;
    for i := 1 step 1 until n do
      if abs((temp[i]−x[i])/x[i]) > relconvg then go to C;
```

```
      converge := converge + 1;
      if converge ≧ 3 then go to TERMINATE else go to D;
C:    converge := 1;
D:    for i := 1 step 1 until n do temp[i] := x[i]
  end m;
  go to THROUGH;
TERMINATE:
  maxit := m;  go to THROUGH;
SING:
  singular := 0;
THROUGH:
  end nonlinearsystem
```

APPENDIX

We include a sample procedure *evaluatekthfunction* for the 2×2 system:

$$\left(1 - \frac{1}{4\pi}\right)(e^{2x_1} - e) + \frac{e}{\pi} x_2 - 2ex_1 = 0$$

$$\frac{1}{2} \sin (x_1 x_2) - \frac{x_2}{4\pi} - \frac{x_1}{2} = 0,$$

one solution of which is $(.5, \pi)$ see [2]

```
procedure evaluatekthfunction (x, y, k);
  integer k;  real y;  array x;
begin switch functionnumber := F1, F2;
  go to functionnumber [k];
F1:  y := 2.71828183 × (.920422528 × (exp(2×x[1]−1)−1)+
        x[2]/3.14159265−2×x[1]);
  go to RETURN;
F2:  y := .5 × sin(x[1]×x[2]) − x[2]/12.5663706 − x[1]/2;
RETURN:
end evaluatekthfunction;
```


References:

1. Brown, K. M. A quadratically convergent method for solving simultaneous non-linear equations. Doctoral Thesis, Dept. Computer Sciences, Purdue U., Lafayette, Ind., Aug., 1966.
2. Brown, K. M., and Conte, S. D. The solution of simultaneous nonlinear equations. Proc. ACM 22nd Nat. Conf., pp 111–114.

ALGORITHM 317*
PERMUTATION [G6]

Charles L. Robinson (Recd. 12 Apr. 1967, 2 May 1967 and 10 July 1967)

Institute for Computer Research, U. of Chicago, Chicago, Ill.

* This work was supported by AEC Contract no. AT (11-1)-614.


```
procedure permute(n, k, v);  value n, k;  integer array v;
    integer n, k;
comment This procedure produces in the vector v the kth
    permutation on n variables. When k = 0, v takes on the value
    1, 2, 3, 4, ··· , n. This algorithm is not as efficient as pre-
    viously published algorithms [1], [2], [3] for generating a
    complete set of permutations but it is significantly better
    for generating a random permutation, a property useful in
    certain simulation applications. Any non-negative value of
    k will produce a valid permutation. To generate a random
    permutation, k should be chosen from the uniform distribu-
    tion over the integers from 0 to n! − 1 inclusive;
begin integer i, q, r, x, j;
  for i := 1 step 1 until n do v[i] := 0;
  for i := n step −1 until 1 do
  begin
q := k ÷ i;  r := k − q × i;  x := 0;  j := n;
no: if v[j] = 0 then
    begin
      if x = r then go to it else x := x + 1
    end;
    j := j − 1;  go to no;
it:  v[j] := i;  k := q;
  end
end
```


References:

1. Coveyou, R. R., and Sullivan, J. G. Algorithm 71, Permutation. *Comm. ACM 4* (Nov. 1961), 497.
2. Peck, J. E. L., and Schrack, G. F. Algorithm 86, Permute. *Comm. ACM 5* (Apr. 1962), 208.
3. Trotter, H. F. Algorithm 115, Perm. *Comm. ACM 5* (Aug. 1962), 434.

Algorithms Policy • Revised August, 1966

A contribution to the Algorithms Department should be in the form of an algorithm, a certification, or a remark. Contributions should be sent in duplicate to the editor, typewritten double spaced. Authors should carefully follow the style of this department with especial attention to indentation and completeness of references.

An algorithm must normally be written in the ALGOL 60 Reference Language [*Comm. ACM 6* (Jan. 1963), 1-17] or in ASA Standard FORTRAN or Basic FORTRAN [*Comm.* ACM 7 (Oct. 1964), 590–625]. Consideration will be given to algorithms written in other languages provided the language has been fully documented in the open literature and provided the author presents convincing arguments that his algorithm is best described in the chosen language and cannot be adequately described in either ALGOL 60 or FORTRAN.

An algorithm written in ALGOL 60 normally consists of a commented procedure declaration. It should be typewritten double spaced in capital and lower-case letters. Material to appear in **boldface** type should be underlined in black. Blue underlining may be used to indicate italic type, but this is usually best left to the Editor. An algorithm written in FORTRAN normally consists of a commented subprogram. It should be typewritten double spaced in the form normally used for FORTRAN or it should be in the form of a listing of a FORTRAN card deck together with a copy of the card deck. Each algorithm must be accompanied by a complete driver program in its language which generates test data, calls the procedure, and produces test answers. Moreover, selected previously obtained test answers should be given in comments in either the driver program or the algorithm. The driver program may be publishedwith the algorithm if it would be of major assistance to a user.

For ALGOL 60 programs, input and output should be achieved by procedure statements, using any of the following eleven procedures (whose body is not specified in ALGOL) [See "Report on Input-Output Procedures for ALGOL 60," *Comm. ACM 7* (Oct. 1964), 628–629]:

insymbol	*inreal*	*outarray*	*ininteger*
outsymbol	*outreal*	*outboolean*	*outinteger*
length	*inarray*	*outstring*	

If only one channel is used by the program for output, it should be designated by 1 and similarly a single input channel should be designated by 2. Examples:

```
outstring (1, 'x=');  outreal (1,x);
for i := 1 step 1 until n do outreal (1,A[i]);
ininteger (2, digit [17]):
```

For FORTRAN programs, input and output should be achieved as described in the ASA preliminary report on FORTRAN and Basic FORTRAN.

It is intended that each published algorithm be well organized, clearly commented, syntactically correct, and a substantial contribution to the literature of Algorithms. It is necessary but not sufficient that a published algorithm operate on some machine and give correct answers. It must also communicate a method to the reader in a clear and unambiguous manner. All contributions will be refereed both by human beings and by an appropriate compiler. Authors should pay considerable attention to the correctness of their programs, since referees cannot be expected to debug them.

Certifications and remarks should add new information to that already published. Readers are especially encouraged to test and certify previously uncertified algorithms. Rewritten versions of previously published algorithms will be refereed as new contributions and should not be imbedded in certifications or remarks.

Galley proofs will be sent to authors; obviously rapid and careful proofreading is of paramount importance.

Although each algorithm has been tested by its author, no liability is assumed by the contributor, the editor, or the Association for Computing Machinery in connection therewith.

The reproduction of algorithms appearing in this department is explicitly permitted without any charge. When reproduction is for publication purposes, reference must be made to the algorithm author and to the *Communications* issue bearing the algorithm.—J.G.Herriot