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COMPLETE SOLUTION OF THE 'EIGHT-PUZZLE'

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'For the last few weeks, the "Fifteen-puzzle" has been prominently before the American Public, and may safely be said to have engaged the attention of nine out of ten persons of both sexes and of all ages and conditions of the community.'

American Journal of Mathematics (1879)

The 'Eight-puzzle' is a reduced form of the 'Fifteen-puzzle', the subject of the somewhat extravagant claim quoted above. Its use in the study of learning processes, both human and programmed, is described in two other papers in this volume, Michie (p. 135) and Doran (p. 105). This paper describes the calculation of optimum solutions to all the 20 160 possible versions of the puzzle.

The 'Eight-puzzle' consists of eight square pieces, numbered 0-7,† capable of sliding in a shallow square tray of size nine times that of the individual pieces: there is thus one empty square. We define a *standard position* as one in which the centre square is empty, and the problem discussed here is that of *sliding* the pieces (i.e., without lifting any piece over another) from one standard position to another in the minimum number of moves.

Without loss of generality, we can obviously select either one fixed starting configuration or one fixed target configuration. For an exhaustive computation, the former is convenient, but in the heuristic programs and experiments described in Michie's and Doran's papers the latter has usually been chosen.

† Piece 0 in this paper corresponds to piece 8 in Michie's and Doran's papers. 0 has been used in the program described below, for ease of representation in 3 binary digits within the computer.

NOTATION

If we regard the squares as being numbered

1	2	3
0		4
7	6	5

then the natural fixed configuration from which to start is that in which piece 0 occupies square 0, etc.

It is convenient to describe permutations of standard positions in terms of numbered squares, which are fixed, rather than numbered pieces, which move. Thus the transition from

5	3	4
0		2
1	7	6

to

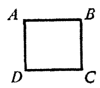
3	4	5
0		2
7	1	6

would be expressed in terms of the disjoint cycles:

(132) (0) (4) (5) (67) or more briefly (132) (67).

(I.e., the piece initially in square 1 goes to square 3, the piece initially in square 3 goes to square 2, the piece initially in square 2 goes to square 1. The pieces in squares 6 and 7 are transposed.)

A *corner twist* is a permutation (starting from a standard position) causing the three pieces nearest to a corner to be permuted cyclically. A clockwise permutation of three pieces is described as twist A, B, C or D depending on the corner concerned:



i.e., A=(012)
B=(234)
C=(456)
D=(670)

Notes. (i) A corner twist is a permutation of order 3: $A^3=I$, the identity, and so $A^2=A^{-1}=(210)$ is an anti-clockwise permutation about the top left-hand corner. There are thus eight possible corner twists: A, B, C, D, A^{-1}, B^{-1}, C^{-1}, D^{-1}.

(ii) A corner twist can be expressed as a pair of transpositions, e.g., (012)=(01) (20), and is an even permutation.

(iii) A corner twist can be carried out in four moves (one piece moves to the centre, two moves round the outside, and the centre piece is moved out). However, as will be shown, a single transposition, being an odd permutation, cannot be carried out within the rules of the puzzle.

It follows from Johnson & Story (1879) that a puzzle can be solved (without lifting pieces over one another) if, and only if, the target position is an even permutation of the starting position. In the case of the 'Eight-puzzle', this may be seen more directly by reference to 'corner twists'. (The arguments below will most easily be followed if a model of the Eight-puzzle is used.)

Theorem 1. In all solvable puzzles, the starting and finishing positions are even permutations of one another.

Proof. Consider any solution which involves no 'back-tracking' (immediate cancellation of a move or series of moves). The same effect can be produced by a succession of corner twists, since, if the fourth move does not complete a corner twist, two dummy moves can now be inserted (one to the centre to complete the twist, the second immediately cancelling the previous move). This process can be continued, along the chosen solution. Hence the solution consists of a product of even permutations.

To demonstrate the converse, consider:

Lemma A. A cyclic permutation of any three adjacent squares is either a corner twist or can be produced by a succession of 4 corner twists.

Proof. If not a corner twist, it must be a permutation along one edge, e.g., (123). This may be expressed:

$$(123)=(321)\,(321)=(12)\,(23)\,(01)\,(01)\,(34)\,(34)\,(12)\,(23)$$

and as disjoint transpositions can be commuted, this is:

$$(12)\,(01)\,(23)\,(34)\,(01)\,(12)\,(34)\,(23)=AB^{-1}\,A^{-1}B$$

and similarly along the other edges.

Note. In fact, the 16-move solution thus obtained is an optimum.

Lemma B. Any transposition is equivalent to a succession of legal 3-cycles (as in Lemma A), followed or preceded by a transposition of adjacent pieces.

The form of this can be seen by the example:

$$(15)=(12)\,(23)\,(34)\,(45)\,(34)\,(23)\,(12)$$

which can be collected *in pairs* to give a succession of legal moves (as in Lemma A), followed or preceded by (12) as required.

Theorem 2. If the starting and finishing positions are even permutations of one another, then the puzzle is solvable.

Proof. Expressing the even permutation as transpositions, it is evidently sufficient to show that a pair of transpositions can be produced legally.

From Lemma B, the first can be expressed as legal moves followed by a transposition of adjacent pieces, and the second as legal moves preceded by transposition of adjacent pieces. If the outstanding transpositions are disjoint, insert dummy pairs of transpositions, e.g.

$(15)\,(47)=$ legal moves . . . $\overline{(12)\,(23)}$ $\overline{(23)\,(34)}$ $\overline{(34)\,(45)}$ . . . legal moves giving a legal method of solution by combining in pairs and using Lemma A.

THE ALTERNATING GROUP

Hence the solvable puzzles correspond to the even permutations of eight elements, that is to the Alternating Group A_8, of order $\frac{1}{2}.8!=20\,160$.

THE 'FIVE-PUZZLE'

Before embarking on a computer program to obtain the minimum path solutions to the eight-puzzle, the present writer wished to obtain an upper bound for the number of moves required in the most difficult cases. This was obtained by first considering the five-puzzle (the eight-puzzle with the bottom row missing, thus losing some symmetry properties), e.g.,

1	2	3
0		4

The previous theorems apply, but the group concerned is the very much smaller Alternating Group A_5, of order $\frac{1}{2}$ $5!=60$.

All paths in this puzzle can be expressed in terms of the four corner twists A, B, A^{-1}, B^{-1}. In general, the number of moves involved is four times the number of twists, but this is reduced if, for example, twist B is followed by A. Given a starting position,

2	3	1
0		4

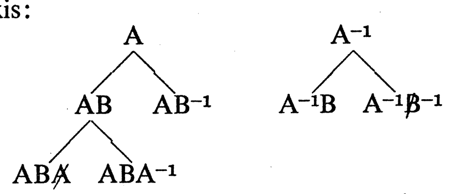
the pieces moved would be 4 1 3 4 and then 4 2 0 4. As the fourth move shifts piece 4 only for the fifth move, to replace it at once, these 8 moves can be reduced to 6—i.e., 4 1 3 2 0 4.

We call this one *elision* and write the path B~~A~~. The same applies when A^{-1} is followed by B^{-1}. The number of moves, m, is given in terms of the number of twists, t, and elisions, e, as: $m=4t-2e$. Hence the path:

$AB^{-1}A^{-1}B$ consists of 16 moves, while
ABAB consists of 14 moves.

Any path between two standard positions can be expressed uniquely in this way.

Of the four possible starting twists, only two, A and A^{-1} say, need be followed, those resulting from an initial B^{-1} or B being deduced by reflection in the vertical axis:



After following these binary trees through 6 twists, all elements of the group A_5 had been generated, the maximum number of moves required being 20. From this a very crude upper bound of 48 moves was deduced for the eight-puzzle.

Before leaving the five-puzzle, it is of interest to note that the group A_5 is isomorphic to the dodecahedral group, and that the axes of symmetry of dodecahedron correspond to the classes of permutation within the five-puzzle as follows:

No. and class of permutations	Axes of dodecahedron
20 cyclic permutations of 3 pieces	10 pairs of opposite vertices (rotations of $2\pi/3$, $4\pi/3$ about each)
24 cyclic permutations of 5 pieces	6 pairs of opposite faces (rotations of $2\pi/5$, $4\pi/5$, $6\pi/5$, $8\pi/5$ about each)
15 pairs of transpositions	15 pairs of opposite edges

SYMMETRY PROPERTIES OF THE 'EIGHT-PUZZLE'

The greater symmetry of the eight-puzzle enables us to carry out an exhaustive trial of all possible paths which might lead to an optimum solution of some puzzle, by starting from one corner twist only (A, say). By rotating the resulting permutation through $\pi/2$, π, $3\pi/2$ and 2π, we reach corresponding positions for which the initial corner twist would have been B, C, D, A respectively, and a subsequent reflection in the vertical axis gives corresponding situations with initial corner twists A^{-1}, D^{-1}, C^{-1}, B^{-1}.

These eight permutations, which are not necessarily distinct, are regarded as forming an *equivalence class*, since a minimum-move solution to any member of the class leads immediately to a minimum-move solution to the remainder. (If X is any given permutation of a standard position, the other members of the equivalence class are given by $\phi_i^{-1}\,X\,\phi_i$ where the ϕ_i are the eight elements of the dihedral group D_4, and the minimum path solutions are obtained by making the same transformations to the corner twists of the solution to the puzzle represented by X.)

The 20 160 possible versions of the eight-puzzle break down into 2572 such equivalence classes, with a consequent saving of computation and storage space.

A further reduction may be made in the number of essentially different puzzles which need be studied, by regarding inverse permutations as belonging to the same equivalence class as one another, thus giving a maximum of 16 members of a class. In this case, the number of equivalence classes reduces to 1439. Although this was done in an early version of the program, it was found less intermediate information needs to be held in the case of the 2572 class version.

ELISION

As with the five-puzzle, the number of moves taken may be less than four times the number of twists, on account of elision. Of the eight twists which might follow a given starting twist (A, say),

(i) Five (B, C, B^{-1}, C^{-1}, D^{-1} in this case) will give no elision.
(ii) One (D) gives one elision,
(iii) Two (A, A^{-1}) will never be used in an efficient solution, since $AA=A^{-1}$ and $AA^{-1}=I$.

PROGRAM FOR THE ATLAS COMPUTER

A program was written for the Atlas computer to calculate the minimum move solutions to the 20 160 possible puzzles. Two dictionaries of permutations were held in the machine:

(i) *Main dictionary.* This had one entry for each possible permutation, and simply stated whether this particular situation had been solved so far, and if so, the length of the minimum path solution. (Only 5 bits needed for each entry.)

(ii) *Short dictionary.* This was built up as solutions were found and contained one representative of each equivalence class, as described in 'Symmetry properties of the "Eight-puzzle"' (p. 129). With each permutation stored, there was a list of corner twists which could be the last twist used in a minimum move solution. This list was increased whenever a given permutation was reached by a new route, using the same number of moves.

STORAGE OF PERMUTATIONS IN MAIN DICTIONARY

For economy in space and to avoid searching down the main dictionary, it is desirable to map the even permutations uniquely on to the integers 0 to 20 159.

If C_i, $i=0, 1, \ldots, 7$ is the number on the piece in square i, then

$$S=\tfrac{1}{2}\sum_{i=0}^{5}\left\{(7-i)!\sum_{j=i+1}^{7}H(C_i-C_j)\right\} \tag{1}$$

gives such a mapping, where $H(t)=\begin{cases}1 \text{ if } t>0\\ 0 \text{ if } t<0\end{cases}$

(In the 'target' ordering given in Fig. 1, $C_i=i$, and the sequence number S is 0, whereas in the position

6	5	4
7		3
1	0	2

all values of $H(C_i-C_j)$ are 1 and so

$$S=\tfrac{1}{2}\sum_{i=0}^{5}\left\{(7-i)!\,(7-i)\right\}=\tfrac{1}{2}\sum_{i=0}^{5}\left\{(8-i)!-(7-i)!\right\}=\tfrac{1}{2}\left\{8!-2!\right\}=20\,159.)$$

GENERAL STRATEGY

The process of exhausting all possibilities was carried out as follows:

1. Set identity permutation as solved in 0 moves (main dictionary).
   Place it as first entry on short dictionary.
   Set all corner twists as solved in 4 moves (main dictionary).
   Place twist A as second entry on short dictionary.
   Set $m=6$.

2. Examine all permutations on short dictionary reached in $(m-4)$ moves and apply all twists which involve no elision. Examine all permutations reached in $(m-2)$ moves and apply twists which do achieve elision.
3. For each permutation produced in stage 2, look at the main dictionary entry (direct look-up in an array using sequence number computed as in equation (1)).
   (i) *If unsolved to date*
      (*a*) In main dictionary, record all members of equivalence class as solved in m moves.
      (*b*) Put one representative of class on short dictionary, together with terminal twist(s) which could be used.
   (ii) *Already solved in m moves*
      Find the representative member in the short dictionary and, if necessary, add to list of possible terminal twists.
   (iii) *Already solved in less than m moves*
      No action.
4. If any new permutations have been found in m or $(m-2)$ moves, increase m by 2 and return to stage 2. Otherwise process is complete.

Using the above strategy, the numbers of permutations which can be achieved in m moves were calculated to be:

No. of moves	Total permutations	No of equivalence classes	No. of extended equivalence classes (counting inverses)
0	1	1	1
4	8	1	1
6	8	1	1
8	40	6	5
10	88	11	6
12	232	29	18
14	556	70	40
16	1 254	159	94
18	2 456	312	167
20	4 020	506	278
22	5 048	638	340
24	4 121	529	298
26	1 902	248	139
28	366	49	39
30	60	12	12
Total	20 160	2 572	1 439

PRINTING OF RESULTS

The printed results were designed for use in experiments described in Michie's (pp. 135-52) and Doran's (pp. 105-23) papers and therefore assume Fig. 1 to be the *target* position. Although one member of each equivalence class would be sufficient, the method of look-up would be tedious and so a complete table was printed. Against each possible puzzle is printed:

(i) the starting twists from which it is possible to achieve a minimum path solution;
(ii) the length of the minimum path;
(iii) a specimen minimum path.

The notation for twists is as described on page 126 except that to economise in space across the page, A^{-1} is printed as W, B^{-1} as X, C^{-1} as Y and D^{-1} as Z. The complete tabulation covers 168 pages, of which a specimen half-page is given opposite, together with an explanation of the method of use.

USE OF TABLES

To identify a particular soluble puzzle, it is only necessary to consider the numbers in 6 of the 8 positions, as the sequence of the remaining two is determined by the condition that the permutation be even. The tables are based on the numbers initially in the squares whose target numbers are 0 1 2 3 4 5. Hence, the puzzle

0	2	5
1		3
7	4	6

would be looked up under the code numbers 102536.

In the tables (see Table 1), the first three of these numbers are given at the head of the half-page, the next three on the left of the corresponding solution. For the above case, we read that the minimum path solution is 20 moves long, that such paths can start with either twist W or Z, and that a specimen solution is WX̸Y̸Z̸AD̸Y.

REFERENCES

Hayes, J. E., Michie, D., Pole, K. E., & Schofield, P. D. A. (1965). A quantitative study of problem-solving using sliding block puzzles: the 'Eight-puzzle' and a modified version of the Alexander Passalong Test. *Experimental Programming Reports*, No. 7 (revised ed.). Edinburgh: Experimental Programming Unit, University of Edinburgh.

Johnson, W. W., & Storey, W. E. (1879). Notes on the '15' puzzle. *Amer. J. Math.*, **2**, 397-404.

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          1 0 2
3 4 5 ) DC...... 16 DC̸B̸WX̸Y̸      3 4 6 ) DC..W..Z 18 DC̸B̸ZW̸X̸Y̸     3 4 7 ) DC...... 20 DC̸B̸DWX̸Y̸
3 5 4 ) D.B..... 16 DC̸B̸ZW̸X̸      3 5 6 ) D....... 14 DC̸B̸WX̸       3 5 7 ) D.....Y. 18 DC̸B̸DWX̸
3 6 4 ) D.B..... 18 DC̸B̸WX̸C      3 6 5 ) D.B...YZ 20 DC̸B̸ZW̸X̸C     3 6 7 ) .C...... 18 CDC̸B̸WX̸
3 7 4 ) ..B..... 14 BA̸XY̸Z̸       3 7 5 ) ..B...Y. 18 BA̸X̸Y̸ZC      3 7 6 ) ..B..... 18 BA̸XY̸ZY
4 3 5 ) D.B.W.Y. 18 DC̸B̸YZ̸W̸X̸    3 3 6 ) D.B.W.Y. 22 DWX̸ZAD̸C̸     4 3 7 ) D.B.W.Y. 22 DC̸B̸ZYZ̸W̸X̸
4 5 3 ) DC...... 20 DC̸B̸CWX̸Y̸     4 5 6 ) .......Z 16 ZW̸DC̸B̸A̸     4 5 7 ) D....... 22 DWZCB̸A̸D̸
4 6 3 ) .C..W... 20 CZW̸DC̸B̸A̸     4 6 5 ) D....... 20 DWZBA̸D̸      4 6 7 ) .......Z 20 ZW̸CDC̸B̸A̸
4 7 3 ) ..B..... 22 BA̸YXY̸Z̸Y      4 7 5 ) ..B..... 18 BA̸CXY̸Z̸      4 7 6 ) D.....Y. 24 DWZYBA̸D̸
5 3 4 ) DC...... 20 DC̸B̸YWX̸Y̸     5 3 6 ) ....W..Z 20 WX̸Y̸Z̸AD̸Y     5 3 7 ) .C...... 22 CDC̸B̸CWZ̸
5 4 3 ) D.B..... 20 DC̸B̸CZW̸X̸     5 4 6 ) ..B..... 20 BDC̸B̸WB      5 4 7 ) ......Y. 22 YBA̸YXY̸Z̸
5 6 3 ) D....... 18 DC̸B̸CWX̸      5 6 4 ) .......Z 20 ZW̸DC̸B̸A̸C    5 6 7 ) D....... 20 DC̸B̸DC̸WX̸
5 7 3 ) ..B..... 18 BA̸YXY̸Z̸      5 7 4 ) DC....Y. 24 DC̸B̸DC̸DWX̸    5 7 6 ) ..B...Y. 22 BA̸CXY̸Z̸C
6 3 4 ) ....W... 16 WX̸Y̸Z̸AD̸      6 3 5 ) D....... 18 DC̸B̸YWX̸     6 3 7 ) ....W... 20 WX̸Y̸Z̸YAD̸
6 4 3 ) D....... 22 DC̸B̸CWX̸C     6 4 5 ) .C.....Z 20 CWX̸Y̸Z̸AD̸    6 4 7 ) D.B..... 24 DC̸B̸DC̸WX̸C
6 5 3 ) ......Y. 20 YWX̸Y̸Z̸AD̸     6 5 4 ) B.B..... 24 DWZBA̸D̸Y     6 5 7 ) .C...... 22 CDC̸B̸YWX̸
6 7 3 ) ..B...Y. 22 BA̸YXY̸Z̸C     6 7 4 ) ..B..... 22 BA̸CXY̸Z̸Y     6 7 5 ) D....... 24 DC̸B̸CDC̸WX̸
7 3 4 ) .C.....Z 22 CB̸A̸YZ̸XY̸Z̸   7 3 5 ) .......Z 20 ZW̸X̸DAZ      7 3 6 ) .......Z 24 ZW̸X̸DAZC
7 4 3 ) .....X.. 20 XAXY̸Z̸X      7 4 5 ) D.B..X.. 24 DXAZAB       7 4 6 ) .....XY. 24 XAXY̸Z̸YX
7 5 3 ) ..B..X.. 24 BA̸YZ̸XY̸Z̸Y    7 5 4 ) .C....Y. 24 CZW̸X̸DAZ     7 5 6 ) D....... 24 DC̸B̸YDC̸WX̸
7 6 3 ) ..B..... 20 BA̸YZ̸WY̸Z̸     7 6 4 ) ..B..... 24 BA̸YZ̸CXY̸Z̸    7 6 5 ) ..B..... 24 BA̸YZ̸YXY̸Z̸
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TABLE 1

Specimen half-page from the computer tabulation of Eight-puzzle solutions.